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FIBRE BEAM-COLUMN MODEL FOR NON-LINEAR ANALYSIS OF R/C FRAMES: PART I. FORMULATION

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SUMMARY

This paper presents a fibre beam-column element for the non-linear static and dynamic analysis of reinforced concrete frames. It is assumed that plane sections remain plane and normal to the longitudinal axis. The effects of shear and bond-slip are, thus, presently neglected. The non-linear hysteretic behaviour of the element derives from the constitutive relations of concrete and reinforcing steel fibres into which each section is divided. The element formulation is flexibility-based and relies on force interpolation functions that strictly satisfy the equilibrium of bending moments and axial force along the element. Since the element does not make use of displacement interpolation functions, an iterative algorithm is needed for the determination of the resisting forces during the element state determination. The proposed algorithm is accurate and stable, even in the presence of strength loss, and is, thus, capable of tracing very well the highly non-linear behaviour of R/C members under cyclic load combinations of bending moment and axial force.

KEY WORDS: RC frames; cyclic behaviour; flexibility method; dynamic response

INTRODUCTION

In recent years non-linear dynamic analysis of three-dimensional structural models is used more and more in the assessment of existing structures in zones of high seismic risk and in the development of appropriate retrofit strategies. In this context alternative modelling strategies can be pursued for the study of the global, regional and local hysteretic response of buildings and bridges under strong ground motions. A suitable classification of modelling strategies is based on the complexity of the model. One can distinguish the following model categories with increasing level of refinement and complexity: (1) *Global models*: in this case the non-linear response of a structure is represented at select degrees of freedom. (2) *Discrete finite element (member) models*: in this case the structure is modelled as an assembly of interconnected frame elements with either lumped or distributed non-linearities. (3) *Microscopic finite element models*: the members and joints of the structure are discretized into several large or small two- or three-dimensional finite elements. While refined finite element models might be suitable for the detailed study of small parts of the structure, such as beam to column joints, frame models are presently the only economical solution for the non-linear dynamic response analysis of structures with several hundred members. Member finite element models are the best compromise between simplicity and accuracy in non-linear seismic response studies and represent the

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simplest class of model that still allows significant insight into the seismic response of members and of the entire structure.

Since the inelastic behaviour of reinforced concrete (RC) frames often concentrates at the ends of girders and columns, an early approach to modelling this behaviour was by means of non-linear springs located at the member ends.[1-3] Among the lumped plasticity constitutive models proposed, some include stiffness degradation in flexure and shear,[4-6] 'pinching' under reversal,[6,7] and fixed end rotations at the beam–column joint interface to simulate the effect of bar pull-out.[8,9] To describe the interaction between axial force and bending moment, Lai *et al.*[10] proposed a fibre hinge model with four steel corner springs and one central concrete spring. The limitations of lumped models are discussed in several correlation studies, such as Charney and Bertero[11] and Bertero *et al.*[12]

A more accurate description of the inelastic behaviour of reinforced concrete members is possible with distributed non-linearity models. The constitutive behaviour of the cross section is either formulated in accordance with classical plasticity theory in terms of stress and strain resultants or is explicitly derived by discretization of the cross section into fibres. Earlier models neglect the coupling between axial force and bending moment.[8,13,14] Roufaiel and Meyer[15] refined the initial model by Meyer *et al.* to include the effect of shear and axial force on the flexural hysteretic behaviour. Darvall and Mendis[16] proposed a similar but simpler model, in which the inelastic deformations at the member ends follow a trilinear moment–curvature relation. Takayanagi and Schnobrich[17] proposed to divide the element into a finite number of non-linear rotational springs connected in series. Filippou and Issa[9] also subdivide the element in different subelements in series, each describing a single effect, such as flexure, shear and pull-out.

The first elements with distributed non-linearity were formulated with the classical stiffness method using cubic Hermitian polynomials to approximate the deformations along the element.[18,19] Shear effects were first included in the model proposed by Bazant and Bhat.[20] Menegotto and Pinto[21] interpolated both section deformations and section flexibilities and accounted for the axial force–bending moment interaction. Recent efforts to develop more robust and reliable reinforced concrete frame elements have shown a trend toward flexibility-based formulations that permit a more accurate description of the force distribution within the element. Mahasuverachai and Powell[22] proposed flexibility-dependent shape functions that are continuously updated during the analysis. Kaba and Mahin[23] presented a flexibility-based fibre element, that was later improved by Zeris and Mahin.[24,25]

The main difficulty with flexibility-based models is their implementation in a finite element program. Existing programs are based on the stiffness method of structural analysis and the element subroutines derive the element forces and stiffness for given nodal displacements. Flexibility-based models proposed to date fail to offer a clear and reliable procedure for their implementation in a general purpose finite element program. Existing formulations lead to violations of the initial equilibrium assumptions. These violations necessitate *ad hoc* corrections that do not, however, preclude serious numerical problems. To overcome this problem Ciampi and Carlesimo[26] proposed a consistent flexibility-based method for formulating frame member models.

This paper refines the method of Ciampi and Carlesimo and applies it to the formulation of a fibre beam–column element. The element formulation is cast in the general framework of the mixed method. The proposed element state determination is based on a non-linear iterative algorithm that always maintains equilibrium within the element and eventually converges to a state that satisfies the element constitutive relation within a specified tolerance. The proposed solution algorithm is particularly suitable for the analysis of the highly non-linear hysteretic behaviour of softening members, such as reinforced concrete columns under varying axial load.

MODEL ASSUMPTIONS

The beam–column element without rigid body modes is shown in Figure 1 in the local reference system $\mathbf{x}, \mathbf{y}, \mathbf{z}$. It is divided into a discrete number of cross sections. These are located at the control points of the numerical integration scheme that is used in the element formulation. The formulation of the beam element is based on the assumption of linear geometry. Plane sections remain plane and normal to the longitudinal axis

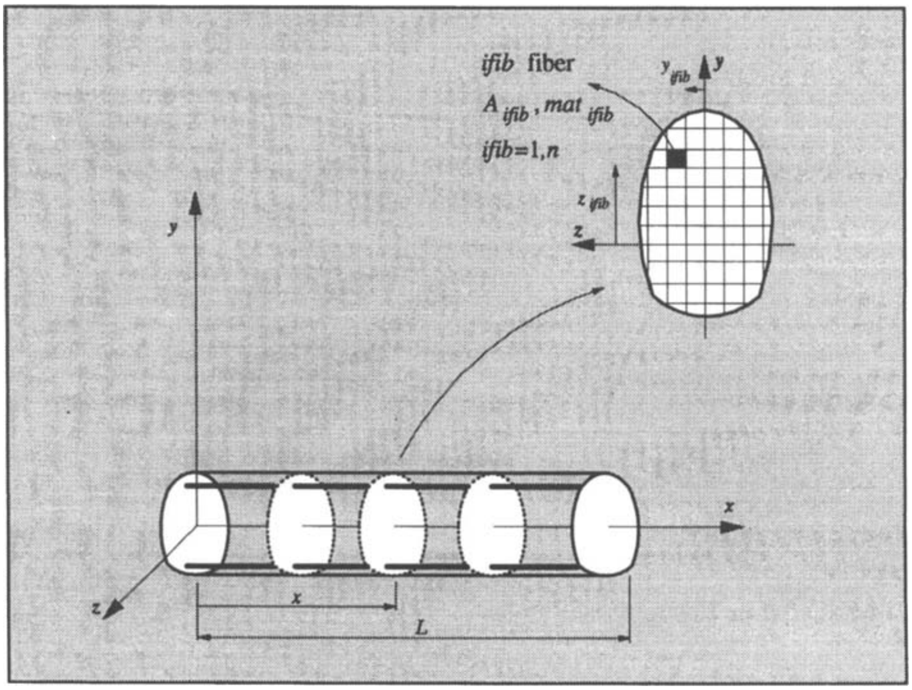

Figure 1. Beam element in the local reference system: subdivision of cross-section into fibres

during the element deformation history. While this hypothesis is acceptable for small deformations of elements composed of homogeneous materials, it does not properly account for phenomena which are characteristic of reinforced concrete elements, such as cracking and bond-slip. The effect of cracking and tension stiffening can be included in the model by an appropriate modification of the stress–strain relation of reinforcing steel or concrete according to the smeared crack concept of finite element analysis. This effect is only significant in the pre-yield phase of response and can be neglected in studies which focus on the hysteretic behaviour under large inelastic deformation reversals. By contrast, the contribution of bond-slip to the element deformations increases with the magnitude and number of loading cycles. The inclusion of bond-slip deformations in an element that is based on section behaviour is a challenging problem, which is beyond the scope of the present study. Shear effects are also neglected, which is a reasonable approximation for medium to large span to depth ratios of the member.

From the assumption that plane sections remain plane and normal to the longitudinal axis, all strains and stresses act parallel to this axis. Since the reference axis is fixed, this implies that the geometric centroids of the sections form a straight line that coincides with the reference axis. If an element does not comply with this hypothesis, it should be divided into subelements that connect the centroids of the selected sections.

Figure 1 shows the section subdivision into fibres. Each section is subdivided into $n(x)$ fibres, where n is a function of x to account for the fact that both cross section and longitudinal reinforcement may vary along the element. It is worth pointing out that the section subdivision into fibres derives from the numerical solution of the integrals for the determination of the section stiffness and resisting force from the corresponding stresses. This point is discussed in more detail in the presentation of the section state determination.

The generalized element forces and deformations and the corresponding section forces and deformations are shown in Figure 2. These are grouped in the following vectors:

Element force vector

$$\mathbf{Q} = \{Q_1 \ Q_2 \ Q_3 \ Q_4 \ Q_5\}^{\mathrm{T}} \tag{1}$$

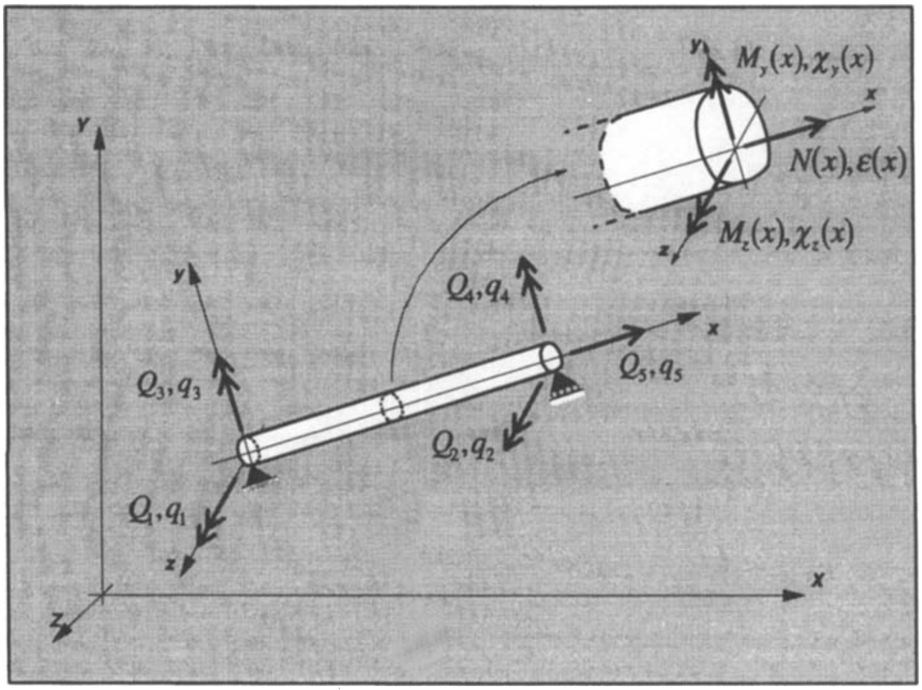

Figure 2. Generalized forces and deformations at the element and section level

Element deformation vector

$$\mathbf{q} = \{q_1 \ q_2 \ q_3 \ q_4 \ q_5\}^{\mathrm{T}} \tag{2}$$

Section force vector

$$\mathbf{D}(x) = \{M_z(x) \ M_y(x) \ N(x)\}^{\mathrm{T}} \tag{3}$$

Section deformation vector

$$\mathbf{d}(x) = \{\chi_z(x) \ \chi_y(x) \ \bar{\varepsilon}(x)\}^{\mathrm{T}} \tag{4}$$

In equation (4) $\chi(x)$ stands for curvature about the respective axis and $\bar{\varepsilon}(x)$ stands for axial strain at the reference axis. From the assumption that plane sections remain plane and normal to the longitudinal axis the fibre strains are related to section deformations with a simple geometric transformation matrix. The element force and deformation vectors in equations (2) and (3), respectively, do not include torsion, since the member behaviour in torsion is assumed to remain linear elastic and uncoupled from the flexural and axial response. While it is rather straightforward to include non-linear torsional response that is uncoupled from flexure and axial force, the coupling of flexure and torsion is a rather complex problem that merits much further study. In most frame structures the torsional response of individual members is not of great significance to the seismic behaviour of the structure.

BEAM–COLUMN ELEMENT FORMULATION

Most studies to date concerned with the non-linear analysis of reinforced concrete frame structures are based on finite element models which are derived with the stiffness method. Recent studies by Kaba and Mahin[23] and Zeris and Mahin[24,25] have demonstrated the advantage of flexibility-based models, but have failed to give a clear and convincing way of determining the element resisting forces from the given displacements.

This difficulty arises when the flexibility-based finite element is implemented in a non-linear analysis program based on the direct stiffness method. In this case the solution of the global equilibrium equations yields the displacements of the structural degrees of freedom. During the state determination phase the resisting forces of all elements in the structure need to be determined. Since in a flexibility-based element there are no deformation interpolation functions to relate the deformations along the element to the end displacements, this process is not straightforward and is not well developed in flexibility-based models proposed to date. This fact has led to some confusion in the numerical implementation of previous models. To overcome this problem Ciampi and Carlesimo[26] proposed a consistent flexibility-based method for formulating frame member models.

This method was refined and applied to the development of two types of elements in Taucer *et al.*[27] and Spacone *et al.*[28] Spacone[29] conducted a large number of static and dynamic simulations of small structures with these elements with great success. The procedure is general in scope and applies to any section material behaviour. The presentation of the state determination process benefits from the derivation of the governing equations according to the two-field mixed method. The element formulation is briefly reviewed here. A more thorough presentation can be found in the studies cited above.

The two-field mixed method uses the integral form of equilibrium and section force–deformation relations to derive the matrix relation between element generalized forces and corresponding deformations. The section force–deformation relation is linearized about the present state and an iterative algorithm is used to satisfy the non-linear section force–deformation relation within the required tolerance. In the following, the steps of the iterative algorithm are denoted by superscript j.

In the two-field mixed method independent interpolation functions are used in the approximation of the deformation and force fields within the element.[30] Denoting with Δ increments of the corresponding quantities, the two incremental fields are written

$$\Delta\mathbf{d}(x) = \mathbf{a}(x)\,\Delta\mathbf{q} \tag{5}$$

$$\Delta\mathbf{D}(x) = \mathbf{b}(x)\,\Delta\mathbf{Q} \tag{6}$$

where matrices $\mathbf{a}(x)$ and $\mathbf{b}(x)$ denote the deformation and force interpolation functions, respectively. In the mixed method formulation the integral forms of equilibrium and section force–deformation relations are expressed first. These are then combined to obtain the matrix relation between element force and deformation increments. The incremental section constitutive relation is linearized according to

$$\Delta\mathbf{d}^{j}(x) = \mathbf{f}^{j-1}(x)\,\Delta\mathbf{D}^{j}(x) + \mathbf{r}^{j-1}(x) \tag{7}$$

where $\mathbf{f}^{j-1}(x)$ and $\mathbf{r}^{j-1}(x)$ are the section flexibility and residual deformations from the previous iteration. The residual deformations can, thus, be interpreted as the linear approximation to the deformation error that arises in the linearization of the section force–deformation relation. The weighted integral form of equation (7) is

$$\int_0^L \delta\mathbf{D}^{\mathrm{T}}(x)[\Delta\mathbf{d}^{j}(x) - \mathbf{f}^{j-1}(x)\,\Delta\mathbf{D}^{j}(x) - \mathbf{r}^{j-1}(x)]\,\mathrm{d}x = \mathbf{0} \tag{8}$$

First, equations (5) and (6) are substituted in equation (8) and observing that equation (8) must hold for arbitrary $\delta\mathbf{Q}$ the integral is equivalent to

$$\mathbf{T}\,\Delta\mathbf{q}^{j} - \mathbf{F}^{j-1}\,\Delta\mathbf{Q}^{j} - \mathbf{s}^{j-1} = \mathbf{0} \tag{9}$$

where $\mathbf{T}$ is a matrix that depends only on the interpolation functions

$$\mathbf{T} = \int_0^L \mathbf{b}^{\mathrm{T}}(x)\,\mathbf{a}(x)\,\mathrm{d}x \tag{10}$$

$\mathbf{F}$ is the element flexibility matrix

$$\mathbf{F} = \int_0^L \mathbf{b}^{\mathrm{T}}(x)\mathbf{f}(x)\mathbf{b}(x)\,\mathrm{d}x \tag{11}$$

and $\mathbf{s}$ is the element residual deformation vector

$$\mathbf{s} = \int_0^L \mathbf{b}^{\mathrm{T}}(x)\mathbf{r}(x)\,\mathrm{d}x \tag{12}$$

Equation (9) is the matrix equivalent of the integral form of the linearized section force–deformation relation. Next, the equilibrium equation of the beam element is satisfied. In the classical two-field mixed method the integral form of the equilibrium equation is derived from the virtual displacement principle:

$$\int_0^L \delta\mathbf{d}^{\mathrm{T}}(x)[\mathbf{D}^{j-1}(x) + \Delta\mathbf{D}^{j}(x)]\,\mathrm{d}x = \delta\mathbf{q}^{\mathrm{T}}\mathbf{Q}^{j} \tag{13}$$

where $\mathbf{Q}^j$ is the vector of nodal forces in equilibrium with the new internal force distribution $\mathbf{D}^{j-1}(x) + \Delta\mathbf{D}^j(x)$. Equations (5) and (6) are substituted in equation (13) and observing that equation (13) must hold for arbitrary $\delta\mathbf{q}$, the integral is equivalent to the following matrix expression:

$$\mathbf{T}^{\mathrm{T}}\mathbf{Q}^{j-1} + \mathbf{T}^{\mathrm{T}}\Delta\mathbf{Q}^{j} = \mathbf{Q}^{j} \tag{14}$$

This is the matrix equivalent of the integral form of the element equilibrium equations. Rearrangement and combination of equations (9) and (14) result in

$$\begin{bmatrix} -\mathbf{F}^{j-1} & \mathbf{T} \\ \mathbf{T}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{Q}^{j} \\ \Delta\mathbf{q}^{j} \end{Bmatrix} = \begin{Bmatrix} \mathbf{s}^{j-1} \\ \mathbf{Q}^{j} - \mathbf{T}^{\mathrm{T}} Q^{j-1} \end{Bmatrix} \tag{15}$$

If the first equation in (15) is solved for $\Delta\mathbf{Q}^j$ and the result is substituted in the second equation, the following expression results:

$$\mathbf{T}^{\mathrm{T}}[\mathbf{F}^{j-1}]^{-1}(\mathbf{T}\Delta\mathbf{q}^{j} - \mathbf{s}^{j-1}) = \mathbf{Q}^{j} - \mathbf{T}^{\mathrm{T}}\mathbf{Q}^{j-1} \tag{16}$$

The selection of the interpolation functions $\mathbf{b}(x)$ and $\mathbf{a}(x)$ for the beam element greatly simplifies the above expression. Matrix $\mathbf{b}(x)$ is readily obtained from the equilibrium of axial forces and bending moments within the element. The selection of $\mathbf{a}(x)$ in the present application of the mixed approach does not affect the element formulation because of the special selection of deformation and force resultants $\mathbf{q}$ and $\mathbf{Q}$, respectively. These measures are conjugate resultants from a work viewpoint, that is the product $\mathbf{q}^{\mathrm{T}}\mathbf{Q}$ is the beam external work. This fact, peculiar to the proposed Bernoulli beam, leads to $\mathbf{T} = \mathbf{I}$, where $\mathbf{I}$ is the 3x3 identity matrix, irrespective of the selection of $\mathbf{a}(x)$. With the simplification $\mathbf{T} = \mathbf{I}$, equation (16) becomes

$$[\mathbf{F}^{j-1}]^{-1}(\Delta\mathbf{q}^{j} - \mathbf{s}^{j-1}) = \Delta\mathbf{Q}^{j} \tag{17}$$

The final equation (17) expresses the linearized matrix relation between the element force increments $\Delta\mathbf{Q}^j$ and the corresponding deformation increments $\Delta\mathbf{q}^j - \mathbf{s}^{j-1}$. The element stiffness matrix is written in the form $[\mathbf{F}]^{-1}$ to stress the fact that it is obtained by inverting the flexibility matrix. The element formulation can also be derived from the force method, but the mixed approach is more general in scope and the resulting matrix equation (17) is closely linked to the element state determination algorithm, since it contains both the imposed element deformation increments $\Delta\mathbf{q}^j$ and the residual deformations $\mathbf{s}^{j-1}$ that arise from the linearization of the non-linear constitutive relation of the section.

ELEMENT STATE DETERMINATION

The implementation of the model in a general purpose finite element analysis program requires the determination of the resisting forces and stiffness matrix that correspond to the displacements at the

structural nodes. In a finite element that is based on the stiffness method of analysis the problem is readily solved. First, the section deformations are obtained directly from the element displacements using deformation interpolation functions. The corresponding section resisting forces and stiffness are subsequently determined from the section force–deformation relation. The weighted integrals of the section resisting forces and stiffness over the element length yield the element resisting forces and stiffness matrix and complete the process of element state determination.

In a flexibility-based element this process is not as straightforward. While the element stiffness matrix is derived by inverting the element flexibility matrix, the element forces present the biggest challenge because they cannot be readily derived from the section forces. The procedure presented in Spacone[29] and Spacone *et al.*[28] introduces an iteration scheme at the element level that is similar to the Newton–Raphson method used for the solution of the equilibrium equations at the structural level. At the structural degrees of freedom, force increments are applied and the Newton–Raphson iterations intend to reduce the unbalanced forces to sufficiently small values at each load step. The solution of these equations at the structural level yields displacement increments at the end nodes of each element. Thus, the iterations during the element state determination phase of the algorithm intend to reduce the deformation residuals to sufficiently small values. Figure 3 shows the relation between element and section state determination during step i of the Newton–Raphson algorithm at the structural degrees of freedom.

At the ith Newton–Raphson iteration it is necessary to determine the element resisting forces for the current element deformations

$$\mathbf{q}^i = \mathbf{q}^{i-1} + \Delta\mathbf{q}^i \tag{18}$$

To this end an iterative process denoted by index j is introduced inside the ith Newton–Raphson iteration. The first iteration corresponds to $j = 1$. The initial state of the element is represented by point A and $j = 0$ in Figure 3. With the initial element tangent stiffness matrix $[\mathbf{F}^{j=0}]^{-1} = [\mathbf{F}^{i-1}]^{-1}$ and the given element deformation increments $\Delta\mathbf{q}^{j=1} = \Delta\mathbf{q}^i$, the corresponding element force increments are

$$\Delta\mathbf{Q}^{j=1} = [\mathbf{F}^{j=0}]^{-1}\Delta\mathbf{q}^{j=1} \tag{19}$$

The section force increments $\Delta\mathbf{D}^{j=1}(x)$ can now be obtained from equation (6) with the force interpolation functions $\mathbf{b}(x)$. With the section flexibility matrix at the end of the previous Newton–Raphson iteration $\mathbf{f}^{j=0}(x) = \mathbf{f}^{i-1}(x)$, the linearization of the section force-deformation relation yields the section deformation increments

$$\Delta\mathbf{d}^{j=1}(x) = \mathbf{f}^{j=0}(x)\Delta\mathbf{D}^{j=1}(x) \tag{20}$$

The section deformations are updated to the state that corresponds to point B in Figure 3 with $\mathbf{d}^{j=1}(x) = \mathbf{d}^{i-1}(x) + \Delta\mathbf{d}^{j=1}(x)$. The section stiffness and resisting forces corresponding to the new deformations need to be computed. This process, called section state determination, is discussed in the following paragraph. It suffices now to point out that the strain distribution $\varepsilon^{j=1}(x, y, z)$ in a section is readily derived from the assumption that plane sections remain plane and normal to the longitudinal axis. The constitutive relations of the concrete and steel fibres of the section yield the corresponding stress $\sigma^{j=1}(x, y, z)$ and tangent modulus $E^{j=1}(x, y, z)$, which are integrated over the cross-sectional area to yield the section resisting forces $\mathbf{D}_{\mathrm{R}}^{j=1}(x)$ and stiffness matrix $\mathbf{k}^{j=1}(x)$. The section stiffness is then inverted to yield the flexibility matrix $\mathbf{f}^{j=1}(x)$ (Figure 3).

The unbalanced forces at the section are the difference between applied and resisting forces

$$\mathbf{D}_{\mathrm{U}}^{j=1}(x) = \mathbf{D}^{j=1}(x) - \mathbf{D}_{\mathrm{R}}^{j=1}(x) \tag{21}$$

The product of unbalanced forces $\mathbf{D}_{\mathrm{U}}^{j=1}(x)$ with the current section flexibility $\mathbf{f}^{j=1}(x)$ yields residual deformations $\mathbf{r}^{j=1}(x)$:

$$\mathbf{r}^{j=1}(x) = \mathbf{f}^{j=1}(x)\mathbf{D}_{\mathrm{U}}^{j=1}(x) \tag{22}$$

The residual deformations, therefore, represent the linear approximation to the deformation error that arises in the linearization of the section force–deformation relation (Figure 3). While any suitable flexibility matrix

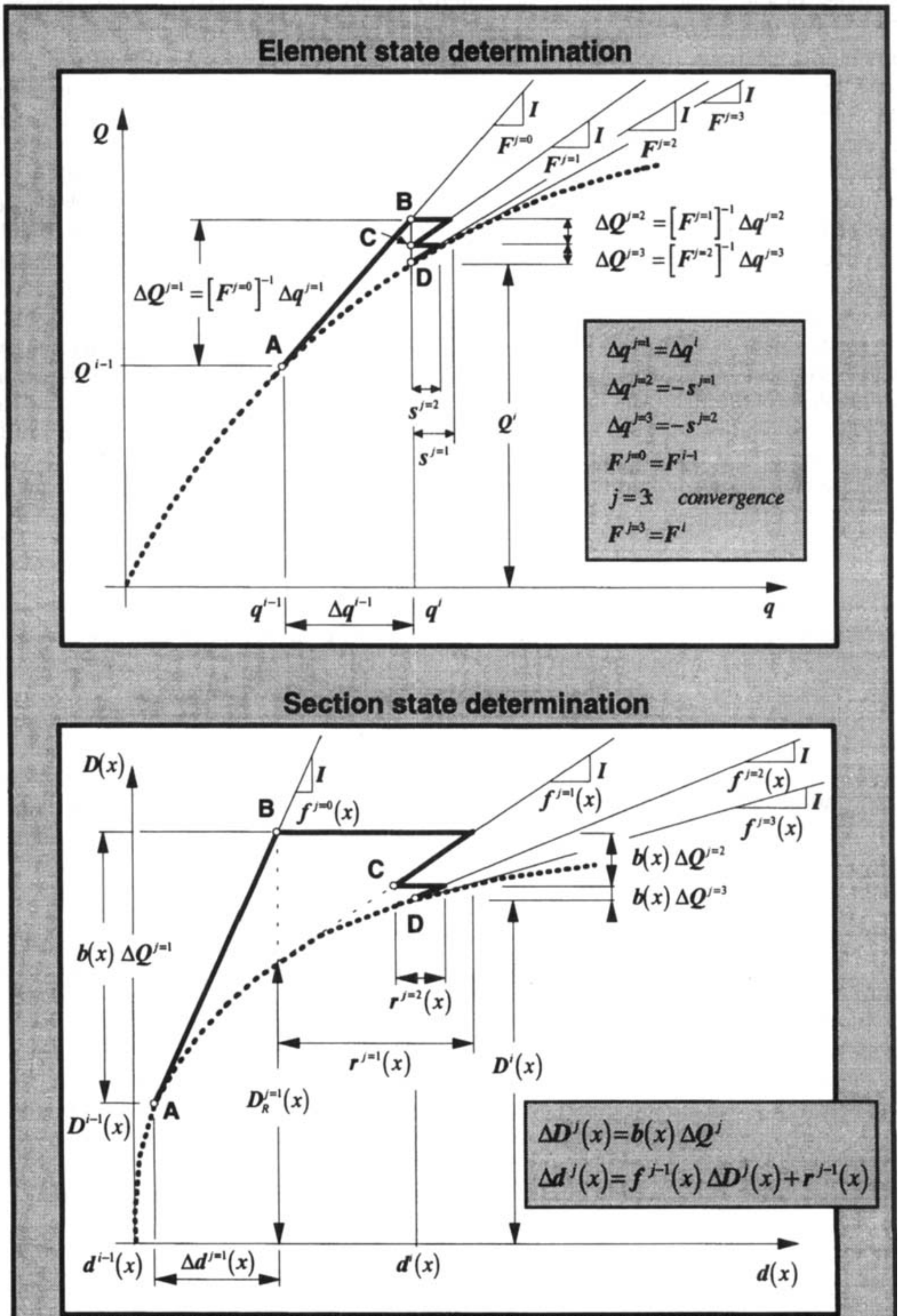

Figure 3. Element and section state determination for flexibility-based element: computation of element resisting forces $\mathbf{Q}^i$ corresponding to element deformations $\mathbf{q}^i$

can be used for the purpose, the tangent flexibility matrix is used in the proposed model, because it offers the fastest convergence rate.

The residual section deformations are integrated over the element length according to the virtual force principle and yield the residual element deformations:

$$\mathbf{s}^{j=1} = \int_0^L \mathbf{b}^{\mathrm{T}}(x)\,\mathbf{r}^{j=1}(x)\,\mathrm{d}x \tag{23}$$

At this point the first iteration $j = 1$ of the corresponding iteration loop is complete. The final element and section states for $j = 1$ correspond to point B in Figure 3.

The residual section deformations $\mathbf{r}^{j=1}(x)$ and corresponding element deformations $\mathbf{s}^{j=1}$ are determined in the first iteration, but the corresponding deformation vectors are not updated. Instead, they form the starting point of the remaining steps within iteration loop j. The existence of residual element deformations $\mathbf{s}^{j=1}$ violates compatibility at the element end nodes, where the deformations should be equal to $\mathbf{q}^i$. Thus, in order to restore compatibility corrective forces equal to $-[\mathbf{F}^{j=1}]^{-1}\mathbf{s}^{j=1}$ must be applied at the element ends, where $\mathbf{F}^{j=1}$ is the updated element tangent flexibility matrix according to equation (11). A corresponding force increment $-\mathbf{b}(x)[\mathbf{F}^{j=1}]^{-1}\mathbf{s}^{j=1}$ is applied at all integration points of the element inducing a deformation increment $-\mathbf{f}^{j=1}(x)\mathbf{b}(x)[\mathbf{F}^{j=1}]^{-1}\mathbf{s}^{j=1}$. Thus, in the second iteration $j = 2$ the state of the element and each control section of the element changes as follows: the element forces are updated to $\mathbf{Q}^{j=2} = \mathbf{Q}^{j=1} + \Delta\mathbf{Q}^{j=2}$ where $\Delta\mathbf{Q}^{j=2} = -[\mathbf{F}^{j=1}]^{-1}\mathbf{s}^{j=1}$. The section forces and deformations are updated to $\mathbf{D}^{j=2}(x) = \mathbf{D}^{j=1}(x) + \Delta\mathbf{D}^{j=2}(x)$ and $\mathbf{d}^{j=2}(x) = \mathbf{d}^{j=1}(x) + \Delta\mathbf{d}^{j=2}(x)$, respectively, where $\Delta\mathbf{D}^{j=2}(x) = \mathbf{b}(x)\,\Delta\mathbf{Q}^{j=2}$ and $\Delta\mathbf{d}^{j=2}(x) = \mathbf{r}^{j=1}(x) + \mathbf{f}^{j=1}(x)\,\Delta\mathbf{D}^{j=2}(x)$.

The state of the element and the sections at the end of the second iteration $j = 2$ correspond to point C in Figure 3. The new section flexibility matrices $\mathbf{f}^{j=2}(x)$and the new residual deformation vectors

$$\mathbf{r}^{j=2}(x) = \mathbf{f}^{j=2}(x)\,\mathbf{D}_{\mathrm{U}}^{j=2}(x) \tag{24}$$

are determined at each control section. The residual section deformations are then integrated to obtain the residual element deformations $\mathbf{s}^{j=2}$ and the new element flexibility matrix $\mathbf{F}^{j=2}$ is determined by integration of the section flexibility matrices $\mathbf{f}^{j=2}(x)$ according to equation (11). This completes the second iteration within loop j. The third and subsequent iterations follow exactly the same scheme. Convergence is achieved when the selected element convergence criterion is satisfied. In Spacone[29] the energy measure $\mathbf{s}^{\mathrm{T}}\mathbf{K}\mathbf{s}$ is used for the purpose and the element iterations converge when

$$\frac{\mathbf{s}^{j^{\mathrm{T}}}\mathbf{K}^j\mathbf{s}^j}{\Delta\mathbf{q}^{1^{\mathrm{T}}}\mathbf{K}^0\,\Delta\mathbf{q}^1} \leqslant \mathrm{tol} \quad (\text{for } j > 1) \tag{25}$$

With the conclusion of iteration loop j the element resisting forces for the given deformations $\mathbf{q}^i$ are established, as represented by point D in Figure 3.

It is important to point out that during the element iterations the end deformations $\mathbf{q}^i$ do not change after the first iteration $j = 1$. While in the first iteration increments $\Delta\mathbf{q}^{j=1} = \Delta\mathbf{q}^i$ are added to the element deformations $\mathbf{q}^{i-1}$ at the end of the previous Newton–Raphson iteration, in the subsequent iterations $j > 1$ only the element forces change until the non-linear algorithm converges to the element resisting forces $\mathbf{Q}^i$ that correspond to element deformations $\mathbf{q}^i$. This is illustrated at the top of Figure 3, where points B, C and D, which represent the state of the element at the end of subsequent iterations in loop j, lie on the same vertical line, while the corresponding points at the control sections of the element do not, as shown at the bottom of Figure 3.

In the proposed non-linear analysis method the element equilibrium is always satisfied in a strict sense during the element state determination, because the section forces are derived from the element forces by force interpolation functions according to equation (6). While element equilibrium is satisfied during each iteration of loop j, the section force–deformation relation and, consequently, the element force–deformation relation is only satisfied within a specified tolerance when convergence is achieved at point D in Figure 3. In other words, during subsequent iterations the element forces approach the value that corresponds to the

imposed element deformations, while strictly satisfying element equilibrium at all times. The linearization and approximation of the force–deformation relation in the proposed non-linear analysis method is preferable to the approximation of either the element equilibrium or the element compatibility conditions, particularly when considering the uncertainty in the definition of material constitutive relations.

SECTION STATE DETERMINATION

Similar to the element state determination, the section state determination involves the calculation of resisting forces $\mathbf{D_R}(x)$ and stiffness matrix $\mathbf{k}(x)$ that correspond to deformations $\mathbf{d}(x)$. From the hypothesis that plane sections remain plane and normal to the longitudinal axis the strain at point (y,z) of cross section x is $\varepsilon(x,y,z) = \mathbf{l}(y,z)\mathbf{d}(x)$, where $\mathbf{l}(y,z)$ is the simple geometric vector

$$\mathbf{l}(y,z) = \{-y \;\; z \;\; 1\} \tag{26}$$

The strain distribution $\varepsilon(x,y,z)$ with the constitutive relations of the constituent materials yields the tangent material modulus $E(x,y,z)$ and stress $\sigma(x,y,z)$. The section stiffness matrix $\mathbf{k}(x)$ and the resisting forces $\mathbf{D_R}(x)$ are then determined with the virtual force principle

$$\mathbf{k}(x) = \int_{A(x)} \mathbf{l}^{\mathrm{T}}(y,z)\, E(x,y,z)\, \mathbf{l}(y,z)\, \mathrm{d}A \tag{27}$$

$$\mathbf{D}(x) = \int_{A(x)} \mathbf{l}^{\mathrm{T}}(y,z)\, \sigma(x,y,z)\, \mathrm{d}A \tag{28}$$

The evaluation of the integrals in a computer program requires the selection of a numerical integration scheme. In the present study a section at location x is subdivided into $\mathrm{n}(x)$ fibres and the midpoint integration rule is used for the integrals in equations (27) and (28). The resulting section stiffness and forces are

$$\mathbf{k}(x) = \sum_{\mathrm{ifib}=1}^{\mathrm{n}(x)} \mathbf{l}^{\mathrm{T}}(x, y_{\mathrm{ifib}}, z_{\mathrm{ifib}})(EA)_{\mathrm{ifib}}\mathbf{l}(x, y_{\mathrm{ifib}}, z_{\mathrm{ifib}}) \tag{29}$$

$$\mathbf{D_R}(x) = \sum_{\mathrm{ifib}=1}^{\mathrm{n}(x)} \mathbf{l}^{\mathrm{T}}(x, y_{\mathrm{ifib}}, z_{\mathrm{ifib}})(\sigma A)_{\mathrm{ifib}} \tag{30}$$

The accuracy of the above integrals depends on the number and location of the fibres. Too few fibres tend to underestimate the section capacity, while too many fibres are computationally expensive. The selection of the optimum number of section fibres is beyond the scope of this paper. The subject is addressed in more detail by Spacone.[29]

It is finally important to point out that the proposed numerical evaluation of the section integrals in equations (27) and (28) by subdivision of the section into fibres and application of the midpoint rule is only one of a number of possible numerical integration methods. The merit of using higher-order schemes, such as Gauss integration formulas, should be explored in future studies in the interest of reducing the number of function evaluations at the section. While the discontinuous nature of stress and stiffness distribution in a reinforced concrete section renders the fibre subdivision and the midpoint rule particularly suitable, higher-order schemes, such as Gauss integration formulas, might offer significant advantages for homogeneous sections in the quest to reduce the number of function evaluations. It is, however, important to point out that nothing in the formulation of the beam–column element hinges on the notion of a fibre subdivision of the cross section, which is just a particularly suitable numerical scheme for the evaluation of the section integrals.

MATERIAL MODELS

The non-linear behaviour of the proposed beam element derives entirely from the material constitutive laws. The validity of the analytical results depends therefore on the accuracy of the material models. Since the

present study is limited to the hysteretic behaviour of reinforced concrete members and the effect of bond-slip is neglected, only two material models are required: one for concrete and one for reinforcing steel. The element formulation simplifies the task of material model selection to uniaxial behaviour, which is thoroughly studied and well established to date. Three-dimensional effects on material behaviour can be included into the uniaxial model by appropriate modification of the parameters that define the monotonic envelope. This is important in the case of concrete, where confinement by transverse and longitudinal reinforcement has a significant effect on the stress–strain behaviour.

The models used in the present study are those discussed in Filippou *et al.*[31] While the reinforcing steel model in Filippou *et al.*[31] remains one of the most accurate and convenient to use, many improved confined concrete models have been proposed in the last ten years[32,33] and an improvement of the original model is implemented in the proposed beam element. The model offers a good compromise between simplicity and accuracy and remains an excellent choice for the hysteretic behaviour of concrete. In any case the modular program architecture that underlies the proposed beam element allows for the easy exchange of material models and a more extensive material library is under development.

It is worth noting that both stress–strain models are explicit functions of strain. This is an important feature of material models in conjunction with the proposed fibre model, where fibre strains are determined from section deformations according to $\varepsilon(x, y, z) = \mathbf{l}(y, z)\mathbf{d}(x)$. The stress determination only involves a function evaluation based on the current fibre stress and strain and the given strain increment. This reduces the computational effort considerably relative to material models that are not explicit functions of strain, such as the well-known Ramberg–Osgood steel model.

The reinforcing steel stress–strain behaviour is shown in Figure 4 and is described by the non-linear model of Menegotto and Pinto,[21] as modified by Filippou *et al.*[31] to include isotropic strain hardening. The model is computationally efficient and agrees very well with experimental results from cyclic tests on reinforcing steel bars. In spite of the simple formulation the original model is capable of reproducing well experimental results. Its major drawback stems from its failure to allow for isotropic hardening. To account for this effect Filippou *et al.*[31] proposed a stress shift in the linear yield asymptote as a function of maximum plastic strain.

As far as the concrete stress–strain behaviour is concerned, there exists some disagreement as to the effect of the concrete model on the hysteretic behaviour of RC members under bending moment reversals with small values of axial force. Some investigators have concluded that a crude concrete model suffices to accurately predict experimental results. This might be true in the case of monotonic loading and cyclic loading that is restricted to small inelastic excursions. It is not true, however, in the case of severe cyclic

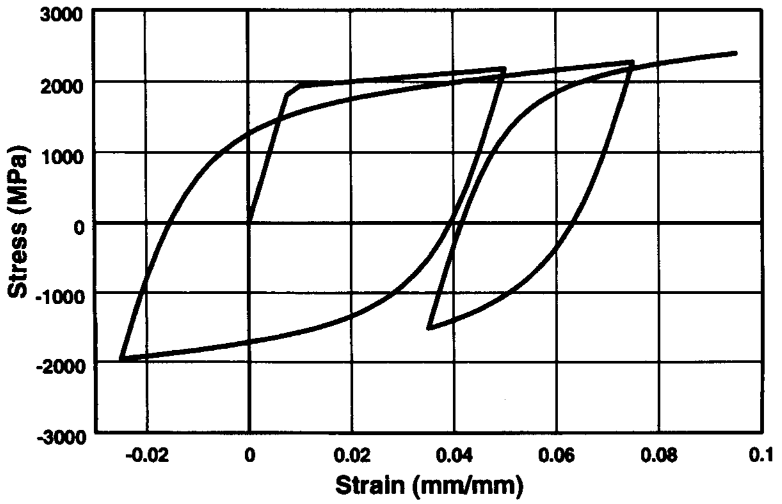

Figure 4. Hysteretic steel stress–strain relation

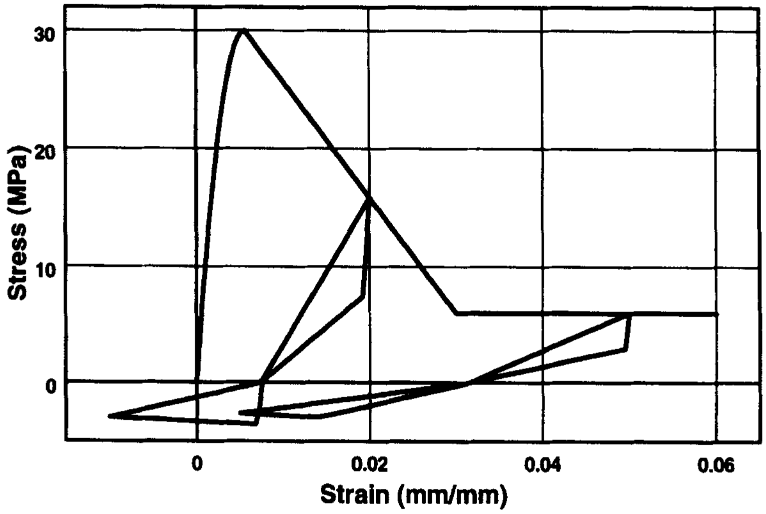

Figure 5. Hysteretic concrete stress–strain relation

loading. The results of the companion paper indicate that the strength deterioration of RC members under large deformation reversals depends largely on the capacity of confined concrete to sustain stresses in the strain range beyond maximum strength. This requires the use of a refined concrete model.

The model used in this study is shown in Figure 5 and is described in more detail by Mohd Yassin.[34] The monotonic envelope curve of concrete in compression follows the model of Kent and Park,[35] as extended later by Scott *et al.*[36] Even though more accurate and complete models have been published since, the modified Kent–Park model represents a good compromise between simplicity and accuracy. The model accounts for the effect of confinement by transverse reinforcement by modifying the strain softening slope of the monotonic envelope in Figure 5. The modified Kent-Park model was enhanced by the inclusion of the tensile behaviour of concrete and, in particular, the effect of tension stiffening. This extension will not be discussed further in this paper, since it is of little relevance to the correlation studies in the companion paper. The hysteretic behaviour of the concrete model includes the gradual degradation of stiffness under unloading and reloading in compression, as shown in Figure 5. More details are provided by Mohd Yassin.[34] The concrete material model does not presently include loss of strength under cyclic load. This is an important effect for columns under cyclic flexure with varying axial load and needs to be addressed in future studies.

NUMERICAL INTEGRATION

The integrals involved in the element formulation, i.e. equations (11) and (12), are evaluated numerically with the Gauss–Lobatto integration scheme. In spread plasticity models this procedure is superior to the classical Gauss integration method because it always includes the end sections of the integration field. When no distributed loads act on the element, the end sections experience the largest forces and undergo the largest inelastic excursions. Monitoring the end sections, therefore, results in improved accuracy of the non-linear response of the beam. By contrast, the outermost integration points of the classical Gauss integration method only approach the ends with increasing order of integration, but never coincide with the end sections and, thus, result in overestimation of the member strength. The mth-order Gauss–Lobatto integration scheme is based on the expression

$$\mathrm{I} = \int_{-1}^{1} g(\xi)\,\mathrm{d}\xi = w_1\, g(\xi_1 = -1) + \sum_{h=2}^{m-1} w_h\, g(\xi_h) + w_m\, g(\xi_m = 1) \tag{31}$$

where h denotes the monitored section and w_h is the corresponding weight.[37] The mth-order Gauss–Lobatto scheme permits the exact integration of polynomials of degree up to $(2m - 3)$.

CONCLUSIONS

The objective of this study is the development of a reliable and computationally efficient beam–column finite element model for the analysis of reinforced concrete members under cyclic load reversals of biaxial bending moments with varying axial force. The non-linear behaviour of the element is monitored at several control sections that are, in turn, discretized into longitudinal steel and concrete fibres. The non-linear section behaviour, thus, derives from the integration of the non-linear stress–strain behaviour of the fibres. This permits the modelling of any type of frame member composed of one or more materials, as long as the assumption of plane sections remaining plane is reasonably accurate. In the case of reinforced concrete members this assumption implies a perfect bond between reinforcing steel and surrounding concrete.

The element formulation is based on force interpolation functions for bending moment and axial force that strictly satisfy element equilibrium and, thus, belongs to the category of flexibility-based elements. It is the first element of its kind to offer a consistent state determination algorithm for the determination of the stiffness matrix and resisting forces for given element end deformations. The element can be, therefore, readily implemented in an existing non-linear analysis program.

The proposed algorithm is general and can be used with any non-linear section force–deformation relation. The procedure involves an element iteration scheme that converges to a state that satisfies the material constitutive relations within the specified tolerance. During the iterations element equilibrium and compatibility are always satisfied in a strict sense by the assumed force interpolation functions and the imposed compatibility condition, respectively.

The proposed model offers three major advantages over traditional stiffness-based beam–column elements: (a) the use of exact force interpolation functions in the element requires fewer elements for the representation of the non-linear behaviour of a structure, (b) no numerical difficulties arise in connection with the possible strength loss and softening of individual sections; and (c) the element can readily incorporate distributed element loads by the addition of the exact internal force distribution function under the given element loads, as discussed in Taucer *et al.*[27]

Finally, the proposed non-linear solution strategy is superior to the event-to-event solution method, whose numerical effort increases with the complexity of the element, the number of fibres in the section description and the number of elements in the structure. Thus, significant savings in computational time should be expected for three-dimensional structures with beam–column elements containing many fibres.

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APPENDIX

Notation

$\mathbf{a}$	deformation interpolation functions
A	fibre area
$\mathbf{b}$	force interpolation functions
$\mathbf{d}$	section deformation vector
$\mathbf{D}$	section force vector
$\mathbf{D}_R$	section resisting force vector

$\mathbf{D}_U$	section unbalanced force vector
E	fibre Young modulus
$\mathbf{f}$	section flexibility matrix
$\mathbf{F}$	element flexibility matrix
$\mathbf{I}$	identity matrix
$\mathbf{k}$	section stiffness matrix
$\mathbf{K}$	element stiffness matrix
$\mathbf{l}$	geometric vector relating section deformations to fibre strains
M_z	section bending moment about local z axis
M_y	section bending moment about local y axis
N	section axial force along local x axis
$q_1, \ldots, q_5$	element generalized deformations
$Q_1, \ldots, Q_5$	element generalized forces
$\mathbf{q}$	element deformation vector
$\mathbf{Q}$	element force vector
$\mathbf{r}$	section residual deformation vector
$\mathbf{s}$	element residual deformation vector
$\mathbf{T}$	matrix depending on interpolation functions only
χ_z	section curvature about local **z** axis
χ_y	section curvature about local **y** axis
$\bar{\varepsilon}$	section strain along local **x** axis
ε	fibre strain
σ	fibre stress

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